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On the Andreadakis conjecture of the automorphism group of a free group

東京理科大学理学部第二部数学科　佐藤 隆夫 * (Satoh, Takao)
Department of Mathematics, Faculty of Science Division II,
Tokyo University of Science

Abstract

In this article, we show that the third subgroup of the Andreadakis-Johnson filtration of the automorphism group of a free group coincides with the the third group of the lower central series of the IA-automorphism group.

The origin of the study of surface automorphisms goes back to the pioneer works by Dehn and Nielsen in early 20th century. In particular, they studied the homeomorphism groups and the mapping class groups of surfaces. In order to describe mapping classes of surfaces, the most basic and standard way is to consider their actions on the homology groups of the surfaces. In most cases, however, such descriptions kill many deep and important deta of surface automorphisms in general. To describe surface automorphisms completely, Nielsen considered to study the actions of the mapping class groups on the fundamental groups of surfaces, and obtained plenty of remarkable results.

In general, the fundamental groups are non-abelian groups, and to deal with non-abelian groups and their automorphisms is sometimes quite complicated, compared to the case of finitely generated abelian groups. In order to investigate the automorphism groups of non-abelian groups, Andreadakis [1] introduced descending filtrations of the automorphism groups of groups. Let G be a group, and $\mathrm{Aut}\,G$ its automorphism group. Let $G = \Gamma_G(1) \supset \Gamma_G(2) \supset \cdots$ be the lower central series of G. The action of $\mathrm{Aut}\,G$ on the $(k+1)$-st nilpotent quotient group $G/\Gamma_G(k+1)$ induces the homomorphism $\mathrm{Aut}\,G \to \mathrm{Aut}\,(G/\Gamma_G(k+1))$. Its kernel is denoted by $\mathcal{A}_G(k)$. Then we have the descending filtration

$$\mathrm{Aut}\,G \supset \mathcal{A}_G(1) \supset \mathcal{A}_G(2) \supset \cdots$$

of $\mathrm{Aut}\,G$. We call this filtration the Andreadakis-Johnson filtration. (We will explain why we attach the name "Johnson" in the next paragraph.) Andreadakis [1] showed that this filtration is central. More precisely, the commutator subgroup of $\mathcal{A}_G(k)$ and $\mathcal{A}_G(l)$ is contained in $\mathcal{A}_G(k+l)$ for any $k, l \geq 1$. Hence, each of the graded quotient $\mathcal{A}_G(k)/\mathcal{A}_G(k+1)$ for $k \geq 1$ is an abelian group. The graded quotients $\mathcal{A}_G(k)/\mathcal{A}_G(k+1)$ are considered to be a sequence of approximations of $\mathrm{Aut}\,G$ by abelian groups, and are

*e-address: takao@rs.tus.ac.jp

one of powerful tools to study the group structure of $\mathrm{Aut}\, G$. If G is finitely generated, then so is $\mathcal{A}_G(k)/\mathcal{A}_G(k+1)$ for any $k \geq 1$. However, to determine the structure of it is quite a difficult problem in general.

Here we remark Johnson's results on mapping class groups of surfaces in the 1980s. The mapping class group of a compact oriented surface with one boundary component can be embedded into the automorphism group of a free group by classical works of Dehn and Nielsen in the 1910s and in early 1920s respectively. Hence we can consider the descending filtration of the mapping class group by restricting the Andreadakis-Johnson filtration to it. This filtration is called the Johnson filtration of the mapping class group, and it became famous among topologists. The first subgroup of the Johnson filtration is called the Torelli group. In the 1980s, Johnson studied the group structure of the Torelli group in a series of works [15], [16], [17] and [18]. In particular, he gave a finite set of generators of the Torelli group, and constructed a certain homomorphism τ to determine the abelianization of it. Today, his homomorphism τ is called the first Johnson homomorphism, and it is generalized to Johnson homomorphisms of higher degrees. Over the last two decades, good progress was made in the study of the Johnson homomorphisms of mapping class groups through the works of many authors including Morita [24, 25, 26], Hain [12] and others. The definition of the Johnson homomorphisms of the mapping class group can be easily generalized to those of $\mathrm{Aut}\, G$ for any group G. To put it plainly, the Johnson homomorphisms are powerful tools to study the graded quotients $\mathcal{A}_G(k)/\mathcal{A}_G(k+1)$ of the Andreadakis-Johnson filtration of $\mathrm{Aut}\, G$. (For details, see our survey papers [36] and [35].)

Now, the first subgroup $\mathcal{A}_G(1)$ is called the IA-automorphism group of G, and usually denoted by $\mathrm{IA}(G)$. The letters I and A stands for "Identity" and "Automorphism" respectively due to Bachmuth [2]. Let $\mathcal{A}'_G(1) \supset \mathcal{A}'_G(2) \supset \cdots$ be the lower central series of $\mathrm{IA}(G)$. Since the Andreadakis-Johnson filtration is central, $\mathcal{A}_G(k)$ contains $\mathcal{A}'_G(k)$ for each $k \geq 1$. Then, it is a natural question to ask: How much is the difference between $\mathcal{A}_G(k)$ and $\mathcal{A}'_G(k)$? Andreadakis focused his interests on the case where G is a free group, and studied the above question.

Let F_n be the free group of rank n with basis $x_1, \ldots, x_n$. Nielsen [27] showed that $\mathrm{IA}(F_2)$ coincides with the inner automorphism group of F_2. In 1935, Magnus [21] showed that $\mathrm{IA}(F_n)$ is finitely generated. The group structure of $\mathrm{IA}(F_n)$ is, however, less well-understood in general at the moment. Krstić and McCool [20] showed that $\mathrm{IA}(F_3)$ is not finitely presentable. For $n \geq 4$, it is not known whether $\mathrm{IA}(F_n)$ is finitely presentable or not. We should remark that Day and Putman [10] obtained an infinite presentation for $\mathrm{IA}(F_n)$ One of reasons why the study of $\mathrm{IA}(F_n)$ has not achieved good progress so much seems that the combinatorial complexity increases quite rapidly as n tends to large.

Andreadakis [1] showed that each of $\mathcal{A}_{F_n}(k)/\mathcal{A}_{F_n}(k+1)$ is free abelian group of finite rank, and gave its rank for $k = 1$ by using Magnus's generators. He also showed that $\mathcal{A}_{F_2}(k) = \mathcal{A}'_{F_2}(k)$ for any $k \geq 1$, and $\mathcal{A}_{F_3}(k) = \mathcal{A}'_{F_3}(k)$ for $1 \leq k \leq 3$. Then he conjectured that $\mathcal{A}_{F_n}(k) = \mathcal{A}'_{F_n}(k)$ for any $n \geq 3$ and $k \geq 1$. Today, this conjecture is called the Andreadakis conjecture. So far, in our previous works [32, 37], we proved that the Andreadakis conjectures restricted to some subgroups are true. However, to attack the original problem is quite difficult due to combinatorial complexities. For any $n \geq 2$, Bachmuth [3] showed that $\mathcal{A}_{F_n}(2) = \mathcal{A}'_{F_n}(2)$. This fact is also obtained from the independent works by Cohen-Pakianathan [7, 8], Farb [11] and Kawazumi [19] who determined the abelianization of IA_n. Pettet [28] showed $\mathcal{A}'_{F_n}(3)$ has at most finite index in $\mathcal{A}_{F_n}(3)$ by using the representation theory of the general linear group. Recently, Bartholdi [4, 5] showed that this conjecture is not true in general. In particular, he showed that

$$\mathcal{A}_3(4)/\mathcal{A}'_3(4) \cong (\mathbf{Z}/2\mathbf{Z})^{\oplus 14} \oplus (\mathbf{Z}/3\mathbf{Z})^{\oplus 3},$$
$$\mathcal{A}_3(5)/\mathcal{A}'_3(5) \cong \mathbf{Z}^{\oplus 3} \oplus (\text{torsions})$$

by using a computer. For a general $n \geq 4$, the conjecture is still open. Here we should remark a remarkable result due to Darné [9]. Recently, he proved that the stable Andreadakis conjecture is true. Namely, the natural map $\mathcal{A}'_{F_n}(k)/\mathcal{A}'_{F_n}(k+1) \to \mathcal{A}_{F_n}(k)/\mathcal{A}_{F_n}(k+1)$ induced from the inclusion is surjective for sufficiently large n. The purpose of the paper is to consider the unstable case, and to improve the above Pettet's result. More precisely, we show the following.

Theorem 1. *For any $n \geq 3$, $\mathcal{A}'_{F_n}(3) = \mathcal{A}_{F_n}(3)$.*

In [31, 36], we showed this type of theorems for some quotient groups of the McCool subgroup of $\mathrm{IA}(F_n)$. The above main theorem is the refinement of these previous results.

Here we recall some results about the Johnson filtration of the mapping class groups. Let $\Sigma_{g,1}$ be the compact oriented surface of genus $g \geq 1$ with one boundary component, and $\mathcal{M}_{g,1}$ its mapping class group. The fundamental group of $\Sigma_{g,1}$ is a free group F_{2g} of rank $2g$. Let $\mathcal{M}_{g,1}(1) \supset \mathcal{M}_{g,1}(2) \supset \cdots$ be the Johnson filtration of $\mathcal{M}_{g,1}$. Namely, if we consider $\mathcal{M}_{g,1}$ as a subgroup of $\mathrm{Aut}\, F_{2g}$ through the Dehn-Nielsen embedding, we have $\mathcal{M}_{g,1}(k) = \mathcal{A}_{F_{2g}}(k) \cap \mathcal{M}_{g,1}$. The subgroup $\mathcal{M}_{g,1}(1)$ is the Torelli group. Let $\mathcal{M}'_{g,1}(1) \supset \mathcal{M}'_{g,1}(2) \supset \cdots$ be the lower central series of the Torelli group. So far, it is known that the Johnson filtration does not coincide with the lower central series of the Torelli group. In fact, Johnson [18] determined the abelianization of $\mathcal{M}_{g,1}(1)$, and showed that it has many direct summands of $\mathbf{Z}/2\mathbf{Z}$ by using the Birman-Craggs homomorphism. From this, it immediately follows that $\mathcal{M}_{g,1}(2) \neq \mathcal{M}'_{g,1}(2)$. In addition to this, we should remark Morita's remarkable result. Morita [23] showed that $(\mathcal{M}_{g,1}(3)/\mathcal{M}'_{g,1}(3)) \otimes_{\mathbf{Z}} \mathbf{Q}$ is not trivial for $g \geq 3$ by using the Casson invariants. Thus, the Andreadakis conjecture restricted the mapping class group never holds for topological reasons.

To the best of our knowledge, it seems that there are very few results on the original Andreadakis conjecture for general $n \geq 3$ and $k \geq 4$. It seems to be important to give a theoretical proof if the conjecture is true. In addition to this, if the conjecture is not true, it also seems to be interesting to describe obstructions in algebraic way with the combinatorial group theory, the representation theory, and so on.

In this section, we introduce a strategy to study the difference between the Andreadakis Johnson filtration and the lower central series of IA_n by using combinatorial group theory. Basically, it is the same method as in our previous work [37] for the lower-triangular automorphism groups of free groups.

The strategy to attack the conjecture

Let $n \geq 3$ and fix it. For $k = 1$ and 2, the conjecture is true. We assume that $\mathcal{A}_n(k) = \mathcal{A}'_n(k)$ for $k \geq 1$. Then we have the homomorphism $\mathrm{gr}^k(\mathcal{A}'_n) \to \mathrm{gr}^k(\mathcal{A}_n)$ induced from the natural inclusion map. If the conjecture is true, it suffices to show that this map is injective. In fact, if this map is injective, we can conclude $\mathcal{A}_n(k+1) = \mathcal{A}'_n(k+1)$ from the fact that $\mathcal{A}_n(k) = \mathcal{A}'_n(k)$ and $\mathcal{A}'_n(k+1) \subset \mathcal{A}_n(k+1)$. In general, however, it is quite difficult to study the structure of $\mathcal{A}'_n(k)/\mathcal{A}'_n(k+1)$ directly. Thus, we use the Johnson homomorphisms. By using our previous result obtained in [30], we determined the cokernel of the composition map

$$\tau'_k : \mathrm{gr}^k(\mathcal{A}'_n) \to \mathrm{gr}^k(\mathcal{A}_n) \to H^* \otimes_{\mathbf{Z}} \mathcal{L}^(_n k+1)$$

for $m \geq k+2$. More precisely, let $\mathcal{C}_n(k)$ be the quotient module of $H^{\otimes k}$ by the action of cyclic group of order k on the components:

$$\mathcal{C}_n(k) := H^{\otimes k} \big/ \langle a_1 \otimes a_2 \otimes \cdots \otimes a_k - a_2 \otimes a_3 \otimes \cdots \otimes a_k \otimes a_1 \,|\, a_i \in H \rangle.$$

In [33], we showed that $\mathrm{Coker}(\tau'_k) = \mathcal{C}_n(k)$ for any $k \geq 2$ and $n \geq k+2$. Furthermore, recently, Darné showed that the natural map $\mathrm{gr}^k(\mathcal{A}'_n) \to \mathrm{gr}^k(\mathcal{A}_n)$ is surjective for $n \geq k+2$. This induces that

$$\mathrm{Coker}(\tau_k) = \mathcal{C}_n(k)$$

for $n \geq k+2$. This means that we can give a lower bound on the number of generators of $\mathrm{gr}^k(\mathcal{A}'_n)$ by considering $\mathrm{rank}_{\mathbf{Z}}(\mathrm{Im}(\tau_k))$. Thus, if we want to give an affirmative answer to the conjecture, it suffices to show that $\mathrm{gr}^k(\mathcal{A}'_n)$ is generated by $\mathrm{rank}_{\mathbf{Z}}(\mathrm{Im}(\tau_k))$ elements.

Let us consider the case where $k = 2$. Pettet [28] determined the $\mathrm{GL}(n, \mathbf{Q})$-module structure of $\mathrm{Im}(\tau'_{2,\mathbf{Q}})$, and gave

$$\begin{aligned} \mathrm{rank}(\mathrm{Im}(\tau_2)) &= \dim_{\mathbf{Q}}(\mathrm{Im}(\tau'_{2,\mathbf{Q}})) \\ &= \frac{1}{3}n^2(n^2-4) + \frac{1}{2}n(n-1) = \frac{1}{6}n(n+1)(2n^2-2n-3). \end{aligned}$$

In this article, for $k = 2$, we show that $\mathrm{gr}^2(\mathcal{A}'_n)$ is generated by the above number of elements for $n \geq 3$.

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